(1.) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2^{n+1}+3^{n-1}}{2^{n+1}+3^{n+2}}\right)^{n}$

Odmocninové kritérium: $\displaystyle\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{2^{n+1}+3^{n-1}}{2^{n-1}+3^{n+2}} =$

$$= \lim_{n\to\infty}\frac{\cancel{3^n}\cdot\left(2\cdot\left(\frac{2}{3}\right)^n+\frac{1}{3}\right)}{\cancel{3^n}\cdot\left(\frac{1}{2}\cdot\left(\frac{2}{3}\right)^n+9\right)} = \frac{0+\frac{1}{3}}{0+9} = \frac{1}{27} \in [0,1)$$

$\Rightarrow$ řada konverguje

$\displaystyle\sum_{n=1}^{\infty}\left(\frac{2^{n+1}}{3^{n+2}}+\frac{n}{3^n}\right)$ řada konverguje, neboť je součtem dvou konvergentních řad

- $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n+1}}{3^{n+2}} = \frac{2}{9}\cdot\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ - geometrická řada s kvocientem $\frac{2}{3} < 1$
- $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^n}$ konverguje podle podílového kritéria

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \lim_{n\to\infty}\frac{\cancel{3^n}}{\cancel{3^n}\cdot 3^1}\cdot\frac{n+1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{3}\cdot\frac{1+\frac{1}{n}}{1} = \frac{1}{3} \in [0,1)$$

$$\sum_{n=1}^{\infty}\left(\sqrt{n^2+7n}-n\right)\cdot\frac{\sqrt{n^2+7n}+n}{\sqrt{n^2+7n}+n} = \sum\frac{(n^2+7n)-n^2}{\sqrt{n^2+7n}+n} =$$

$$= \sum\frac{7\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{7}{n}}+1\right)} = \sum\underbrace{\frac{7}{\sqrt{1+\frac{7}{n}}+1}}_{\xrightarrow{n\to\infty}\ \frac{7}{2}}$$

Řada diverguje, neboť obecný člen $\xrightarrow{n\to\infty} \frac{7}{2}$ nemá limitu 0, tedy není splněna nutná podmínka konvergence řady: $\lim_{n\to\infty} a_n \neq 0$

(2.) $$\sum_{n=1}^{\infty} \underbrace{\frac{2^{3n+1}+2^{3n-1}}{3^{2n}+3^{2n+1}}} = \frac{5}{8}\cdot\sum_{n=1}^{\infty}\left(\frac{8}{9}\right)^n = \frac{5}{8}\cdot\frac{\frac{8}{9}}{1-\frac{8}{9}} =$$

$$\frac{2^{3n}\cdot\left(2+\frac{1}{2}\right)}{3^{2n}\cdot(1+3)} = \frac{\frac{5}{2}}{4}\cdot\frac{8^n}{9^n} = \frac{5}{8}\cdot\left(\frac{8}{9}\right)^n$$

$$= \frac{5}{8}\cdot\frac{\frac{8}{9}}{\frac{1}{9}} = \frac{5}{8}\cdot 8 = \underline{\underline{5}}$$

(3.) $$\lim_{n\to\infty}\left(\frac{n+3}{n+2}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^{n+2-2} =$$

$$= \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^{n+2}}_{=e}\cdot\underbrace{\lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^{-2}}_{(1+0)^{-2}=1} = \underline{\underline{e}}$$

(4.) $$a_{n+2}-a_{n+1}-6a_n=0$$

$$x^2-x-6=0$$

$$(x-3)(x+2)=0$$

$$x=3 \vee x=-2$$

F.S.: $3^n, (-2)^n$

$$a_n = c_1\cdot 3^n + c_2\cdot(-2)^n$$

$$a_0 = c_1 + c_2 = 0$$

$$a_1 = 3c_1 - 2c_2 = 1$$

$$c_1 = -c_2 \qquad 3c_1 + 2c_1 = 1$$

$$5c_1 = 1$$

$$c_1 = \frac{1}{5}$$

$$c_2 = -\frac{1}{5}$$

$$a_n = \frac{1}{5}\cdot 3^n - \frac{1}{5}\cdot(-2)^n, \ n\in\mathbb{N}$$

5. $y'' - y' - 6y = x^2$, $y(0)=0$, $y'(0)=1$

I. $y'' - y' - 6y = 0$

$\lambda^2 - \lambda - 6 = 0$

$(\lambda-3)(\lambda+2) = 0$

$\lambda = 3 \vee \lambda = -2$

F.S. e^{3x}, e^{-2x}

řešení homogenní rovnice $y_H = c_1 \cdot e^{3x} + c_2 \cdot e^{-2x}$, $c_1, c_2 \in R$

II. $y_P = Ax^2 + Bx + C = -\frac{1}{6}x^2 + \frac{1}{18}x - \frac{7}{108}$

$2A - (2Ax + B) - 6(Ax^2 + Bx + C) = x^2$

$2A - 2Ax - B - 6Ax^2 - 6Bx - 6C = x^2$

$-6A = 1 \Rightarrow \boxed{A = -\frac{1}{6}}$

$-2A - 6B = 0 \Rightarrow \boxed{B = \frac{1}{18}}$

$2A - B - 6C = 0$

$-\frac{1}{3} - \frac{1}{18} - 6C = 0$

$-\frac{7}{18} = 6C \Rightarrow \boxed{C = -\frac{7}{108}}$

obecné řešení: $y = y_H + y_P = c_1 \cdot e^{3x} + c_2 \cdot e^{-2x} - \frac{1}{6}x^2 + \frac{1}{18}x - \frac{7}{108}$

$y(0) = 0$: $c_1 + c_2 - \frac{7}{108} = 0$

$y' = 3c_1 e^{3x} - 2c_2 e^{-2x} - \frac{1}{3}x + \frac{1}{18}$

$y'(0) = 3c_1 - 2c_2 + \frac{1}{18} = 1$

$c_1 + c_2 = \frac{7}{108} \quad |\cdot 2$

$3c_1 - 2c_2 = \frac{17}{18}$

$5c_1 = \frac{7}{54} + \frac{17}{18} = \frac{58}{54}$

$c_1 = \frac{29}{135} \quad c_2 = -\frac{3}{20}$

řešení: $\boxed{y = \frac{29}{135} \cdot e^{3x} - \frac{3}{20} e^{-2x} - \frac{1}{6}x^2 + \frac{1}{18}x - \frac{7}{108}}$