1.

$$f(x) = \frac{4x}{x^2+1} \qquad D_f = \mathbb{R}$$

x	1	-1	0	$\sqrt{3}$	$-\sqrt{3}$
$f(x)$	2	-2	0	$\sqrt{3}$	$-\sqrt{3}$

$$f'(x) = \frac{4\cdot(x^2+1) - 4x\cdot 2x}{(x^2+1)^2} = \frac{-4x^2+4}{(x^2+1)^2} = (-4)\cdot\frac{x^2-1}{(x^2+1)^2}$$

$$f'(x) = 0 \iff x = \pm 1$$

$f'<0$ ↘ -1 ↗ $f'>0$ 1 ↘ $f'<0$

$$f(1) = \frac{4\cdot 1}{1^2+1} = 2$$

$$f(-1) = \frac{4\cdot(-1)}{(-1)^2+1} = -2$$

lok. maximum: $[1; 2]$

lok. minimum: $[-1; -2]$

$$f''(x) = \left((-4)\cdot\frac{x^2-1}{(x^2+1)^2}\right)' = (-4)\cdot\frac{2x(x^2+1)^2 - (x^2-1)\cdot 2x\cdot 2(x^2+1)}{(x^2+1)^4}$$

$$= (-4)\cdot\frac{\cancel{(x^2+1)}\left(2x(x^2+1) - 4x(x^2-1)\right)}{(x^2+1)^{\cancel{4}\,3}} = (-4)\cdot\frac{2x^3+2x-4x^3+4x}{(x^2+1)^3}$$

$$= (-4)\cdot\frac{-2x^3+6x}{(x^2+1)^3} = 8x\cdot\frac{x^2-3}{(x^2+1)^3} = \frac{8x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}$$

$$f''(x) = 0 \iff x = 0 \lor x = \pm\sqrt{3}$$

$f''<0$ ∩ $-\sqrt{3}$ ∪ $f''>0$ 0 ∩ $f''<0$ $\sqrt{3}$ ∪ $f''>0$

$$f(0) = 0$$

$$f(\sqrt{3}) = \frac{4\sqrt{3}}{(\sqrt{3})^2+1} = \sqrt{3}$$

$$f(-\sqrt{3}) = \frac{4\cdot(-\sqrt{3})}{(-\sqrt{3})^2+1} = -\sqrt{3}$$

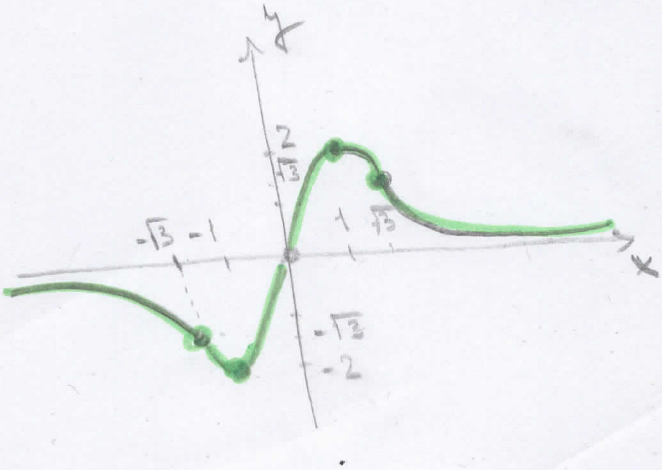
inflexní body: $[0, 0]$, $[\sqrt{3}, \sqrt{3}]$, $[-\sqrt{3}, -\sqrt{3}]$

$$\lim_{x\to\pm\infty} \frac{4x}{x^2+1} \overset{\text{L.P.}}{\underset{\text{"}\frac{\infty}{\infty}\text{"}}{=}} \lim \frac{4}{2x} = 0$$

vodorovná asymptota $y = 0$

$$f(-x) = \frac{4\cdot(-x)}{(-x)^2+1} = -\frac{4x}{x^2+1} = -f(x)$$

$\Rightarrow$ funkce je lichá



Obor hodnot $H_f = \langle -2, 2 \rangle$.

(2a) $\int_{-10}^{10} \frac{4x}{x^2+1}\,dx = 2\cdot\int_{101}^{101} \frac{1}{t}\,dt = 2\cdot\left[\ln|t|\right]_{101}^{101} = \underline{\underline{0}}$

$\left| \begin{array}{l} x^2+1 = t \\ 2x\,dx = dt \end{array} \right|$

(2b) $\int_{\pi}^{2\pi} x\cos x\,dx = \left[x\cdot\sin x\right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi} \sin x\,dx =$

$\left| \begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{array} \right|$

$= 2\pi\cdot\underbrace{\sin 2\pi}_{0} - \pi\cdot\underbrace{\sin\pi}_{0} - \left[-\cos x\right]_{\pi}^{2\pi} = 0-(-1-1) = \underline{\underline{2}}$

(2c) $\int \frac{x^2+13}{x^3-6x^2+13x}\,dx \overset{(*)}{=} \int \frac{1}{x}\,dx + \int \frac{6}{x^2-6x+13}\,dx =$

$\frac{x^2+13}{x^3-6x^2+13x} = \frac{x^2+13}{x(x^2-6x+13)} = \frac{A}{x} + \frac{Bx+C}{x^2-6x+13} \quad \Big|\cdot x(x^2-6x+13)$

$x^2+13 = A(x^2-6x+13) + (Bx+C)\cdot x$

$\underline{x=0}:\quad 13 = A\cdot 13 \Rightarrow \boxed{A=1}$

$x^2+13 = x^2-6x+13 + Bx^2 + Cx$

I. $1 = 1+B \Rightarrow \boxed{B=0}$

II. $0 = -6+C \Rightarrow \boxed{C=6}$

$= \ln|x| + 3\cdot \operatorname{arctg}\frac{x-3}{2} + C$

$\int \frac{6}{x^2-6x+13}\,dx = 6\cdot\int \frac{1}{(x-3)^2+4}\,dx = 6\cdot\int \frac{1}{4\left(\left(\frac{x-3}{2}\right)^2+1\right)}\,dx$

$= \frac{3}{\cancel{2}}\cdot\cancel{2}\operatorname{arctg}\left(\frac{x-3}{2}\right) = 3\operatorname{arctg}\frac{x-3}{2}$

(2d) $$\int_0^1 \sqrt{x}\,(2x-1)^2\,dx = \int_0^1 \sqrt{x}\,(4x^2-4x+1)\,dx =$$

$$= \int_0^1 \left(4x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + x^{\frac{1}{2}}\right) dx = 4\cdot\left[\frac{2}{7}x^{\frac{7}{2}}\right]_0^1 - 4\cdot\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^1$$

$$+\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = 4\cdot\frac{2}{7} - 4\cdot\frac{2}{5} + \frac{2}{3} = 2\cdot\left(\frac{4}{7} - \frac{4}{5} + \frac{1}{3}\right)$$

$$= 2\cdot\frac{60-84+35}{105} = 2\cdot\frac{11}{105} = \underline{\underline{\frac{22}{105}}}$$

(3.) $$f(x,y) = \sqrt{y-x+3}$$

$$z_0 = f(1,2) = \sqrt{2-1+3} = \sqrt{4} = 2$$

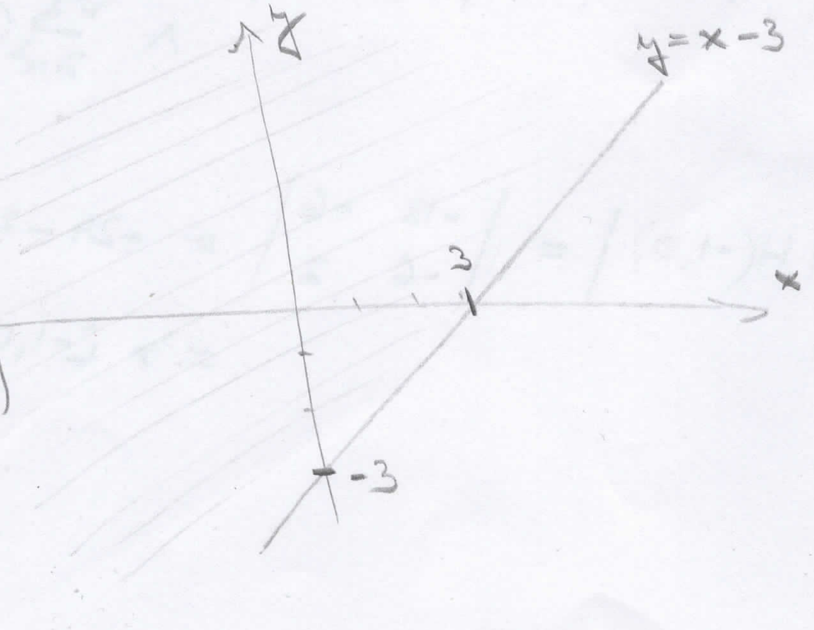
$$\frac{\partial f}{\partial x} = (-1)\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{y-x+3}}\bigg|_{[1,2]} = -\frac{1}{4}$$

$$\frac{\partial f}{\partial y} = 1\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{y-x+3}}\bigg|_{[1,2]} = \frac{1}{4}$$

tečná rovina: $$\boxed{z-2 = \left(-\frac{1}{4}\right)(x-1) + \frac{1}{4}\cdot(y-2)}$$

$$y-x+3 \geq 0$$
$$y \geq x-3$$

$$D_f = \{(x,y)\in\mathbb{R}^2;\ y \geq x-3\}$$



4. $f(x,y) = 2x^3 - 6xy - 6x + y^2 - 6y + 7$

$$\frac{\partial f}{\partial x} = 6x^2 - 6y - 6 = 0 \quad \longleftrightarrow \quad \boxed{y = x^2 - 1}$$

$$\frac{\partial f}{\partial y} = -6x + 2y - 6 = 0 \quad |:2 \quad \longleftarrow \text{dosazení}$$

$$-3x + x^2 - 1 - 3 = 0$$

$$x^2 - 3x - 4 = 0$$

$$(x-4)(x+1) = 0$$

$$x = 4 \quad \vee \quad x = -1$$

$$y = 15 \qquad y = 0$$

stacionární body: $[4; 15]$, $[-1; 0]$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 12x \\ \frac{\partial^2 f}{\partial y^2} &= 2 \\ \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} &= -6 \end{aligned}\right\}$$

Hessova matice

$$H(x,y) = \begin{pmatrix} 12x & -6 \\ -6 & 2 \end{pmatrix}$$

$$|H(4,15)| = \begin{vmatrix} 48 & -6 \\ -6 & 2 \end{vmatrix} = 96 - 36 = 60 > 0$$

$$\wedge \ \frac{\partial^2 f}{\partial x^2}(4,15) > 0$$

$\Longrightarrow [4; 15]$ lok. minimum

$$|H(-1,0)| = \begin{vmatrix} -12 & -6 \\ -6 & 2 \end{vmatrix} = -24 - 36 < 0$$

$\Longrightarrow [-1, 0]$ je sedlový bod