a) $y'' + y' - 6y = 2x + 3$, $y(0) = 1$, $y'(0) = 0$

$$\mathcal{L}(y'' + y' - 6y) = \mathcal{L}(y'') + \mathcal{L}(y') - 6\mathcal{L}(y) =$$

$$= p^2 \cdot \mathcal{L}(y) - p \cdot \underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} + p \cdot \mathcal{L}(y) - \underbrace{y(0)}_{1} - 6\mathcal{L}(y)$$

$$= p^2 \cdot \mathcal{L}(y) - p + p \cdot \mathcal{L}(y) - 6\mathcal{L}(y) - 1$$

$$= \mathcal{L}(y) \cdot (p^2 + p - 6) - p - 1$$

$$\mathcal{L}(2x+3) = \int_0^\infty (2x+3) \cdot e^{-px}\, dx = \left[(2x+3)\left(-\frac{1}{p} e^{-px}\right)\right]_0^\infty$$

$$+ \int_0^\infty \frac{2}{p} e^{-px}\, dx$$

PER-PARTES : $\left| \begin{array}{ll} u = 2x+3 & v' = e^{-px} \\ u' = 2 & v = -\frac{1}{p} e^{-px} \end{array} \right.$

$$\overset{p>0}{=} \underbrace{\lim_{x \to \infty} \left(-\frac{2x+3}{p e^{px}}\right)}_{0} + \frac{3}{p} + \frac{2}{p} \cdot \underbrace{\left[-\frac{1}{p} e^{-px}\right]_0^\infty}_{0 + \frac{1}{p}}$$

$$= \frac{3}{p} + \frac{2}{p^2}$$

$$\Rightarrow \mathcal{L}(y) \cdot (p^2 + p - 6) = p + \frac{3}{p} + \frac{2}{p^2} + 1$$

$$\mathcal{L}(y) = \frac{p^3 + 3p + 2 + p^2}{p^2 \cdot (p^2 + p - 6)}$$

$$y(x) = \mathcal{L}^{-1}\left(\frac{p^3+3p+2+p^2}{p^2(p+3)(p-2)} \right) = \sum_{p\in M} \operatorname{res}_p \left(e^{pt} \cdot \frac{p^3+3p+2+p^2}{p^2(p+3)(p-2)} \right)$$

$$M = \{0; -3; 2\}$$

$$= \frac{4}{5} e^{2t} + \frac{34}{45} e^{-3t} - \frac{t}{3} - \frac{5}{9}$$

$$\operatorname{res}_2 \left(e^{pt} \cdot \frac{p^3+3p+2+p^2}{p^2(p+3)(p-2)} \right) = e^{2t} \cdot \frac{20}{20} \cdot \underbrace{\operatorname{res}_2 \left(\frac{1}{p-2} \right)}_{=1}$$

$$= 1 \cdot e^{2t}$$

$$\operatorname{res}_{-3} \left(\quad -''- \quad \right) = e^{-3t} \cdot \frac{-25}{9 \cdot (-5)} \cdot \underbrace{\operatorname{res}_{-3} \left(\frac{1}{p+3} \right)}_{=1}$$

$$= e^{-3t} \cdot \frac{5}{9}$$

$$\operatorname{res}_0 \left(-''- \right) = \lim_{p \to 0} \left(e^{pt} \cdot \frac{p^3+3p+2+p^2}{(p+3)(p-2)} \right)' =$$

$$= \lim_{p \to 0} \left(t e^{pt} \frac{p^3+3p+2+p^2}{(p+3)(p+2)} + e^{pt} \cdot \frac{(3p^2+3+2p)(p+3)(p-2) - (p^3+3p+2+p^2)(2p+1)}{\left((p+3)(p-2)\right)^2} \right) = -\frac{1}{3} t + \frac{-18-2}{36}$$

$$= -\frac{t}{3} - \frac{5}{9}$$

b) $y'' + y' - 2y = 7x + 8 \qquad y(1) = 0,\ y'(1) = 2$

substituce: $\boxed{x = t + 1} \implies t = x - 1$

$$y'' + y' - 2y = 7t + 15 \ \wedge\ y(0) = 0,\ y'(0) = 2$$

$$\mathcal{L}(y'' + y' - 2y) = \mathcal{L}(y'') + \mathcal{L}(y') - 2\mathcal{L}(y) =$$

$$= p^2\mathcal{L}(y) - p\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{2} + p\mathcal{L}(y) - \underbrace{y(0)}_{0} - 2\mathcal{L}(y)$$

$$= \mathcal{L}(y)\cdot(p^2 + p - 2) - 2$$

$$\mathcal{L}(7x + 15) = \int_0^\infty (7x + 15)e^{-px}\,dx = 7\cdot\int_0^\infty xe^{-px}\,dx + 15\underbrace{\int_0^\infty e^{-px}\,dx}_{\left[-\frac{1}{p}e^{-px}\right]_0^\infty}$$

$$\left|\begin{array}{l} px = z \\ p\,dx = dz \end{array}\right|$$

$$\overset{p>0}{=} 7\cdot\frac{1}{p^2}\underbrace{\int_0^\infty z e^{-z}\,dz}_{=1 \text{ neboť }} + 15\cdot\left(0 + \frac{1}{p}\right)$$

$$\boxed{\int_1^\infty x^n e^{-x}\,dx = n!}$$

$$= \frac{7}{p^2} + \frac{15}{p}$$

$$\implies \mathcal{L}(y)\cdot(p^2 + p - 2) - 2 = \frac{7}{p^2} + \frac{15}{p}$$

$$\mathcal{L}(y) = \frac{7 + 15p + 2p^2}{p^2(p^2 + p - 2)}$$

$$y = \mathcal{L}^{-1}(-\,\text{''}\,-) = \sum_{a\in M} \operatorname{res}_a\left(e^{pt}\cdot\frac{2p^2 + 15p + 7}{p^2(p+2)(p-1)}\right)$$

$$M = \{0; -2; 1\}$$

$$\operatorname{res}_1\left(e^{pt}\,\frac{2p^2+15p+7}{p^2(p+2)(p-1)}\right) = e^{t}\cdot\frac{24}{3}\cdot\underbrace{\operatorname{res}_1\left(\frac{1}{p-1}\right)}_{=1}$$

$$= \underline{8e^{t}} = \underline{\underline{8e^{x-1}}}$$

$t = x-1$

$$\operatorname{res}_{-2}\left(e^{pt}\,\frac{2p^2+15p+7}{p^2(p+2)(p-1)}\right) = e^{-2t}\cdot\frac{(-15)}{4\cdot(-3)}\cdot\underbrace{\operatorname{res}_{-2}\left(\frac{1}{p+2}\right)}_{=1}$$

$$= \frac{5}{4}e^{-2t} = \underline{\underline{\frac{5}{4}e^{-2x+2}}}$$

$$\operatorname{res}_0\left(e^{pt}\,\frac{2p^2+15p+7}{p^2(p+2)(p-1)}\right) = \lim_{p\to 0}\left(e^{pt}\cdot\frac{2p^2+15p+7}{p^2+p-2}\right)'$$

0 je pól násobnosti 2

$$\operatorname{res}_a f(z) = \frac{1}{(k-1)!}\lim_{z\to a}\left(f(z)\cdot(z-a)^{k}\right)^{(k-1)}$$

kde k je násobnosť pólu a

$$= \lim_{p\to 0}\left(t e^{pt}\cdot\frac{2p^2+15p+7}{p^2+p-2} + e^{pt}\cdot\frac{(4p+15)(p^2+p-2) - (2p^2+15p+7)\cdot(2p+1)}{(p^2+p-2)^2}\right) = -\frac{7}{2}t + \frac{-30-7}{4}$$

$$= -\frac{7}{2}(x-1) - \frac{37}{4} = \underline{\underline{-\frac{7}{2}x - \frac{23}{4}}}$$

Záver: $y(x) = \sum_{a\in M}\operatorname{res}_a(\ ) = \underline{\underline{8e^{x-1} + \frac{5}{4}e^{2-2x} - \frac{7}{2}x - \frac{23}{4}}}$