(3b) $f(x) = \frac{3x-2}{2x^2}$, $D_f = \mathbb{R} \setminus \{0\}$ $\lim\limits_{x \to 0_+} f(x) = \lim\limits_{x \to 0_-} f(x) = -\infty$

průsečíky s osami: $[\frac{2}{3}, 0]$

$\lim\limits_{x \to \infty} f(x) = 0$, $\lim\limits_{x \to -\infty} f(x) = 0$ $x = \frac{2}{3} \iff y = 0$

1. derivace: $f'(x) = \frac{3 \cdot 2x^2 - (3x-2) \cdot 4x}{4x^4} = \frac{6x^2 - 12x^2 + 8x}{4x^4} = \frac{8x - 6x^2}{4x^4}$

$f'(x) = 0 \iff 8x - 6x^2 = 0 \quad |:2$

$4x - 3x^2 = 0$

$x(4-3x) = 0 \iff \underline{x=0} \lor \underline{x = \frac{4}{3}}$

$\searrow \quad \nearrow \quad \searrow$

$f'<0 \quad 0 \quad f'>0 \quad \frac{4}{3} \quad f'<0$

$\forall x \in (-\infty, 0) \cup (\frac{4}{3}, \infty): f'(x) < 0$

$\Rightarrow$ funkce je klesající

	$-\infty$ … 0	0 … $\frac{4}{3}$	$\frac{4}{3}$ … ∞
x	$-$	$+$	$+$
$4-3x$	$+$	$+$	$-$
	$-$	$+$	$-$

$\forall x \in (0, \frac{4}{3}): f'(x) > 0 \Rightarrow$ funkce je rostoucí

lokální maximum je v $x = \frac{4}{3}$, $f(x) = \frac{3 \cdot \frac{4}{3} - 2}{2 \cdot \frac{16}{9}} = \frac{2}{2 \cdot \frac{16}{9}} = \frac{9}{16}$

tj. $[\frac{4}{3}; \frac{9}{16}]$ je lok. max.

2. derivace: $f''(x) = \frac{(8-12x)4x^4 - (8x-6x^2) \cdot 16x^3}{16x^8} = \frac{(8-12x)x - (8x-6x^2) \cdot 4}{4x^5} =$

$= \frac{8x - 12x^2 - 32x + 24x^2}{4x^5} = \frac{-24x + 12x^2}{4x^5} = \frac{-6+3x}{x^4}$

$f''(x) = 0 \iff -6 + 3x = 0$

$x = 2$ $f(2) = \frac{3 \cdot 2 - 2}{8} = \frac{1}{2}$

$\Rightarrow$ inflexní bod $[2; \frac{1}{2}]$

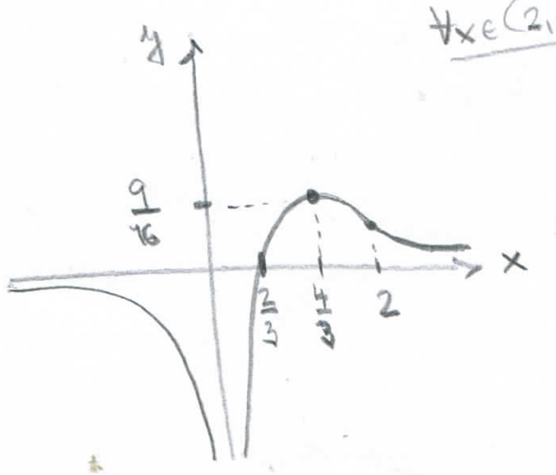
$\cap \quad \cap \quad \cup$

$f''<0 \quad 0 \quad f''<0 \quad 2 \quad f''>0$

$\forall x \in (-\infty, 0) \cup (0, 2): f''(x) < 0 \Rightarrow$ funkce je KONKÁVNÍ

$\forall x \in (2, \infty): f''(x) > 0 \Rightarrow$ funkce je KONVEXNÍ

Graf:



Obor hodnot $H_f = (-\infty, \frac{9}{16}\rangle$

3c $f(x)=\dfrac{x^2}{x-1}$ $\qquad D_f = R\setminus\{1\}$ $\qquad$ L'HOSPITALOVO PRAVIDLO "$\frac{\infty}{\infty}$"

$\left(=\lim\limits_{x\to\infty}\frac{2x}{1}\right)$

$$\lim_{x\to 1_+} f(x) = \frac{1}{0_+} = +\infty \qquad \lim_{x\to\infty}\frac{x^2}{x-1} = \infty$$

$$\lim_{x\to 1_-} f(x) = \frac{1}{0_-} = -\infty \qquad \lim_{x\to -\infty}\frac{x^2}{x-1} = -\infty$$

1. derivace: $f'(x) = \dfrac{2x(x-1)-x^2}{(x-1)^2} = \dfrac{2x^2-2x-x^2}{(x-1)^2} = \dfrac{x^2-2x}{(x-1)^2}$

$$f'(x)=0 \iff x^2-2x=0$$
$$x(x-2)=0$$
$$x=0 \vee x=2 \quad \text{stacionární body}$$

$\nearrow \quad \searrow \quad \nearrow$

$f'>0$ $\quad 0 \quad$ $f'<0$ $\quad 2 \quad$ $f'>0$

$\forall x\in(-\infty,0)\cup(2,\infty)$: $f'(x)>0 \Rightarrow$ funkce roste

$\forall x\in(0,2)$: $f'(x)<0 \Rightarrow$ funkce klesá

lok. maximum: $[0;0]$

lok. minimum: $[2;4]$

2. derivace: $f''(x) = \dfrac{(2x-2)(x-1)^2-(x^2-2x)\cdot 2(x-1)}{(x-1)^4} = \dfrac{(2x-2)(x-1)-2(x^2-2x)}{(x-1)^3}$

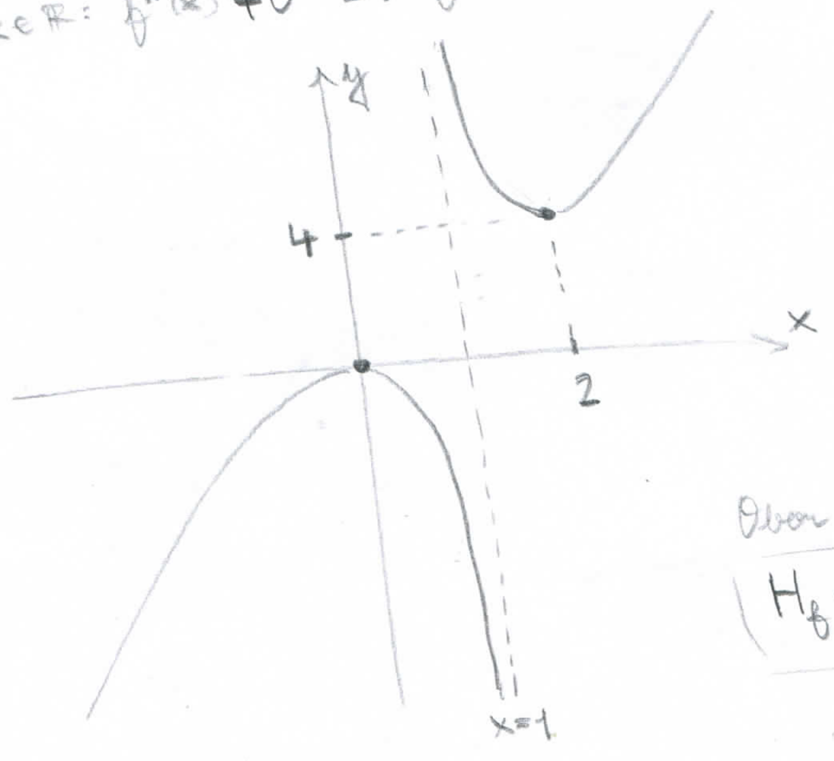
$$= \frac{\cancel{2x^2}-\cancel{2x}-\cancel{2x}+2-\cancel{2x^2}+\cancel{4x}}{(x-1)^3} = \frac{2}{(x-1)^3}$$

$\forall x\in(-\infty,1)$: $f''(x)<0 \Rightarrow$ funkce je KONKÁVNÍ

$\forall x\in(1,\infty)$: $f''(x)>0 \Rightarrow$ funkce je KONVEXNÍ

$\forall x\in R$: $f''(x)\neq 0 \Rightarrow$ funkce nemá inflexní body

Graf:



Obor hodnot:

$H_f = (-\infty,0\rangle \cup \langle 2,\infty)$

(3d) $f(x) = \operatorname{arctg}\left(\frac{1}{x}\right)$ $\quad D_f = \mathbb{R} \setminus \{0\}$

$\lim\limits_{x \to 0_+} f(x) = \arctan\left(\frac{1}{0_+}\right) = \frac{\pi}{2}$ $\quad \lim\limits_{x \to \infty} f(x) = \arctan 0 = 0$

$\lim\limits_{x \to 0_-} f(x) = \arctan\left(\frac{1}{0_-}\right) = -\frac{\pi}{2}$ $\quad \lim\limits_{x \to -\infty} f(x) = \arctan 0 \doteq 0$

$\forall x \in D_f:$ $\arctan\left(\frac{1}{(-x)}\right) = -\arctan\left(\frac{1}{x}\right) \Rightarrow f$ je lichá funkce

1. DERIVACE:

$f'(x) = \frac{1}{\left(\frac{1}{x}\right)^2 + 1} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{1+x^2} \neq 0 \Rightarrow f$ nemá extrém

$\forall x \in D_f: f'(x) < 0 \Rightarrow$ funkce je klesající na D_f

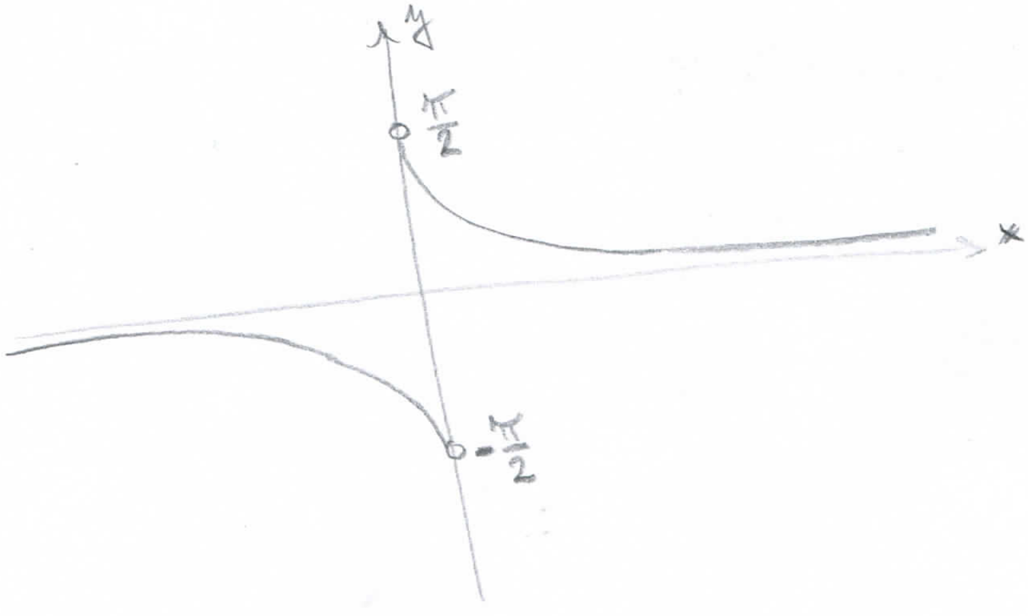
2. DERIVACE:

$f''(x) = \frac{2x}{(1+x^2)^2}$ $\quad f''(x) = 0 \iff x = 0$

$\forall x \in (0, \infty): f''(x) > 0 \Rightarrow$ funkce je konvexní

$\forall x \in (-\infty, 0): f''(x) < 0 \Rightarrow$ funkce je konkávní

Graf:



Obor hodnot: $\boxed{H_f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

$f(x) = \operatorname{arccotg} \frac{x-2}{x}$ $\qquad D_f = \mathbb{R} \setminus \{0\}$

$$\lim_{x \to 0_+} \operatorname{arccotg} \frac{x-2}{x} = \operatorname{arccotg} \frac{(-2)}{0_+} = \pi$$

$$\lim_{x \to 0_-} \operatorname{arccotg} \frac{x-2}{x} = \operatorname{arccotg} \frac{(-2)}{0_-} = 0$$

$$\lim_{x \to +\infty} \operatorname{arccotg} \frac{x-2}{x} = \operatorname{arccotg} \left(\lim_{x \to +\infty} \left(1 - \frac{2}{x}\right) \right) = \operatorname{arccotg} 1 = \frac{\pi}{4}$$

$$\lim_{x \to -\infty} \operatorname{arccotg} \frac{x-2}{x} = \operatorname{arccotg} \left(\lim_{x \to -\infty} \left(1 - \frac{2}{x}\right) \right) = \operatorname{arccotg} 1 = \frac{\pi}{4}$$

$$f'(x) = -\frac{1}{\left(\frac{x-2}{x}\right)^2 + 1} \cdot \frac{x-(x-2)}{x^2} = -\frac{2}{(x-2)^2 + x^2} = -\frac{2}{x^2 - 4x + 4 + x^2} =$$

$$= -\frac{1}{x^2 - 2x + 2} < 0 \quad \forall x \in D_f, \text{ neboť } \forall x \in \mathbb{R}: x^2 - 2x + 2 > 0$$

$\forall x \in D_f : f'(x) < 0 \implies$ funkce f je klesající

$$f''(x) = \frac{2x-2}{(x^2-2x+2)^2}$$

$\cap \quad \cap \quad \cup$

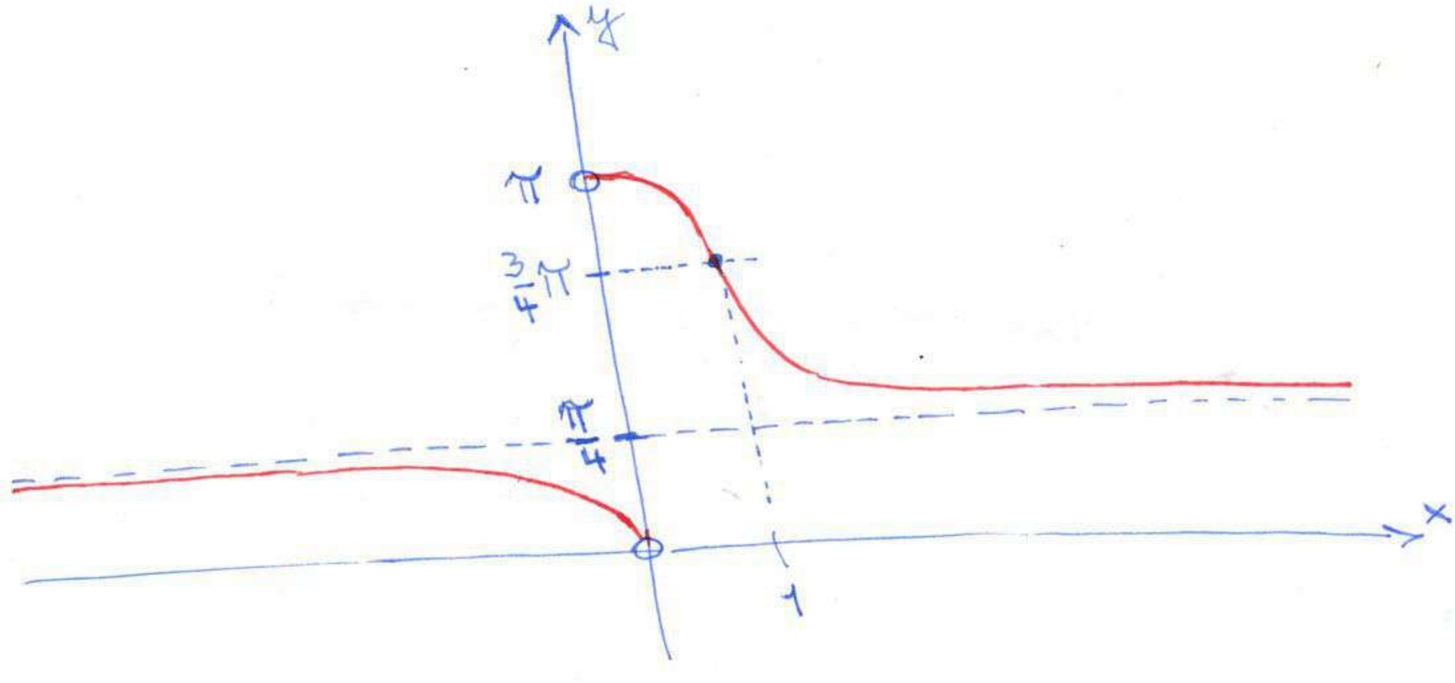
$f'' < 0 \quad 0 \quad f'' < 0 \quad 1 \quad f'' > 0$

$f''(x) = 0 \iff 2x - 2 = 0$

$\underline{x = 1}$ - inflexní bod

$\forall x \in (-\infty, 0) \cup (0, 1) : f''(x) < 0 \implies$ funkce f je konkávní

$\forall x \in (1, \infty) : f''(x) > 0 \implies$ funkce f je konvexní



$$f(x) = \frac{x^2+x-2}{4-2x}$$

x	1	-2	0	4
$f(x)$	0	0	$-\frac{1}{2}$	$-\frac{9}{2}$

$D_f = \mathbb{R} \setminus \{2\}$

průsečíky s osami: $y=0 \iff x^2+x-2=0$

$$(x+2)(x-1)=0$$

$$\underline{x=-2} \vee \underline{x=1}$$

$P_y = [0; -\frac{1}{2}]$

$P_{x_1} = [1; 0]$

$P_{x_2} = [-2; 0]$

$$f'(x) = \frac{(2x+1)(4-2x)-(x^2+x-2)\cdot(-2)}{(4-2x)^2} = \frac{6x+4-4x^2+2x^2+2x-4}{(4-2x)^2}$$

$$= \frac{8x-2x^2}{(4-2x)^2} = \frac{2x\cdot(4-x)}{(4-2x)^2}$$

$f'<0$ | 0 | $f'>0$ | 2 | $f'>0$ | 4 | $f'<0$

$f'(x)=0 \iff \underline{x=0} \vee \underline{x=4}$

lok. minimum: $[0; -\frac{1}{2}]$

lok. maximum: $[4; -\frac{9}{2}]$

$$f(4) = \frac{16+4-2}{4-8} = -\frac{9}{2}$$

$$f''(x) = \left(\frac{8x-2x^2}{(4-2x)^2}\right)' = \frac{(8-4x)(4-2x)^2-(8x-2x^2)\cdot 2(4-2x)(-2)}{(4-2x)^4}$$

$$= \frac{\cancel{(4-2x)}\left((8-4x)(4-2x)+4\cdot(8x-2x^2)\right)}{(4-2x)^{\cancel{4}\,3}}$$

$$= \frac{32-16x-16x+\cancel{8x^2}+32x-\cancel{8x^2}}{(4-2x)^3} = \frac{32}{(4-2x)^3}$$

$\forall x\in(2,+\infty)$: $f''(x)<0 \Rightarrow$ funkce je konkávní

$\forall x\in(-\infty,2)$: $f''(x)>0 \Rightarrow$ funkce je konvexní

$\forall x\in D_f$: $f''(x)\neq 0 \Rightarrow f$ nemá inflexní bod

$\cup$ | $\cap$

$f''>0$ | 2 | $f''<0$

šikmá asymptota:

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\frac{x^2+x-2}{4-2x}}{x} = \lim_{x\to\pm\infty} \frac{x^2+x-2}{4x-2x^2}$$

$$\overset{L.P.}{=} \lim_{x\to\pm\infty} \frac{2x+1}{4-4x} \overset{L.P.}{=} \lim_{x\to\pm\infty} \frac{2}{-4} = \underline{\underline{-\frac{1}{2}}}$$

$$q = \lim_{x\to\pm\infty} \left(f(x) - kx\right) = \lim \left(\frac{x^2+x-2}{4-2x} + \frac{1}{2}x \right)$$

$$= \frac{1}{2} \cdot \lim \left(\frac{x^2+x-2}{2-x} + x \right) = \frac{1}{2} \lim \frac{x^2+x-2+2x-x^2}{2-x}$$

$$= \frac{1}{2} \lim_{x\to\pm\infty} \frac{3x-2}{2-x} \overset{L.P.}{=} \frac{1}{2} \cdot \lim \frac{3}{(-1)} = \underline{\underline{-\frac{3}{2}}}$$

$\Rightarrow$ přímka $\boxed{y = -\frac{1}{2}x - \frac{3}{2}}$ je šikmá asymptota v $\pm\infty$

$$f(x) = \frac{(x+2)(x-1)}{2(2-x)}$$

$x+2$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$2-x$	$+$	$+$	$+$	$-$
	$+$	$-$	$+$	$-$

$-\infty \quad -2 \quad 1 \quad 2 \quad \infty$

$\forall x \in (-\infty, -2) \cup (1, 2): f(x) > 0$

$\forall x \in (-2, 1) \cup (2, \infty): f(x) < 0$

Obor hodnot: $\boxed{H_f = (-\infty, -\frac{9}{2}\rangle \cup \langle -\frac{1}{2}, \infty)}$