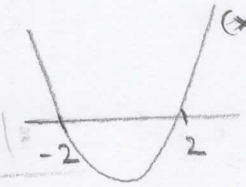
1. $f(x) = \sqrt{x^4 - 4x^2} - x^2$

$x^4 - 4x^2 \geq 0$

$x^2(x^2-4) \geq 0$

$x^2(x-2)(x+2) \geq 0$




$\boxed{D = (-\infty, -2\rangle \cup \langle 2, \infty) \cup \{0\}}$

$\lim\limits_{x \to 2^+} f(x) = f(2) = -4$

$\lim\limits_{x \to 2^-} f(x) = f(2) = -4$

$\forall x \in D_f: f(-x) = f(x) \Rightarrow$ tj. $f$ je sudá $\Rightarrow \lim\limits_{x\to\infty} f(x) = \lim\limits_{x \to -\infty} f(x)$

$$\lim_{x\to\infty}\left(\sqrt{x^4-4x^2} - x^2\right) = \lim_{x\to\infty} \frac{x^4 - 4x^2 - x^4}{\sqrt{x^4-4x^2} + x^2} =$$

$$= \lim_{x\to\infty} \frac{-4\cancel{x^2}}{\cancel{x^2}\left(\sqrt{1 - \frac{4}{x^2}} + 1\right)} \stackrel{A.L.}{=} \frac{(-4)}{\sqrt{1+0}+1} = -\frac{4}{2} = \underline{\underline{-2}}$$

3. $f(x) = -2x^2 + 2x + 40 = (-2)(x^2 - x - 20) = (-2)(x-5)(x+4)$

| $x$ | 5 | -4 | 0 | $\frac{1}{2}$ | -6 |
|---|---|---|---|---|---|
| $f(x)$ | 0 | 0 | 40 | $\frac{81}{2} = 40{,}5$ | -44 |

vrchol: $\frac{5+(-4)}{2} = \frac{1}{2}$

$f(\frac{1}{2}) = (-2)(-\frac{9}{2}) \cdot \frac{9}{2} = \frac{81}{2}$

$f'(x) = -4x + 2$

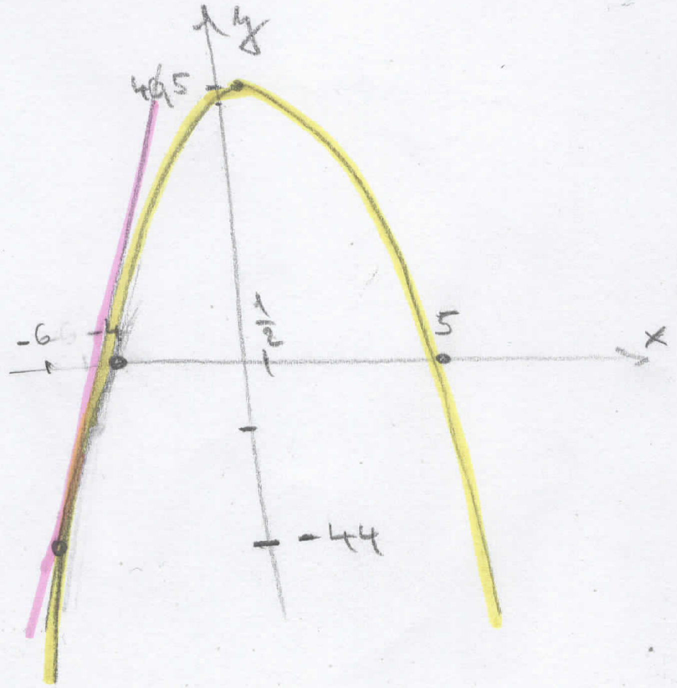
$f'(-6) = 26$

tečna: $y = 26x + q$

$-44 = 26 \cdot (-6) + q$

$q = 156 - 44 = 112$

t: $\boxed{y = 26x + 112}$



2.

$$\lim_{x\to 1} \frac{\ln(6x^2-5x)}{x^2+2x-3} \overset{\text{L.P.}}{=} \lim_{x\to 1} \frac{\frac{1}{6x^2-5x}\cdot(12x-5)}{2x+2} =$$

$$\overset{\text{A.L.}}{=} \frac{\frac{1}{6-5}\cdot(12-5)}{4} = \frac{7}{4}$$

4.

$$f(x) = \frac{x^2+3}{x^2+1} \qquad D_f = \mathbb{R} \quad \text{neboť} \quad \forall x\in\mathbb{R}: x^2+1 \neq 0$$

$$f'(x) = \frac{2x(x^2+1)-(x^2+3)\cdot 2x}{(x^2+1)^2} = \frac{2x-6x}{(x^2+1)^2} = -\frac{4x}{(x^2+1)^2}$$

$$f'(x) = 0 \iff x = 0$$

$$\nearrow \qquad \searrow$$

$f'>0 \quad 0 \quad f'<0$

$$f(0) = \frac{0^2+3}{0^2+1} = 3 \qquad \text{lok. maximum: } [0;3]$$

$$f''(x) = -\frac{4(x^2+1)^2-4x\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} = -\frac{\cancel{(x^2+1)}\left(4(x^2+1)-16x^2\right)}{(x^2+1)^{\cancel{4}\,3}}$$

$$= -\frac{4-12x^2}{(x^2+1)^3}$$

$$f''(x) = 0 \iff 4-12x^2 = 0 \quad |:4$$

$$1-3x^2 = 0$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\frac{1}{\sqrt{3}}$$

$\cup \qquad \cap \qquad \cup$

$f''>0 \quad -\frac{1}{\sqrt{3}} \quad f''<0 \quad \frac{1}{\sqrt{3}} \quad f''>0$

$$f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{\frac{1}{3}+3}{\frac{1}{3}+1} = \frac{\frac{10}{3}}{\frac{4}{3}} = \frac{5}{2}$$

INFLEXNÍ BODY: $\left[\pm\frac{1}{\sqrt{3}};\frac{5}{2}\right]$

$\forall x\in\left(-\infty,-\frac{1}{\sqrt{3}}\right)\cup\left(\frac{1}{\sqrt{3}},\infty\right): f''(x)>0 \Rightarrow f$ je konvexní

$\forall x\in\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right): f''(x)<0 \Rightarrow f$ je konkávní

$$\lim_{x\to\pm\infty} \frac{x^2+3}{x^2+1} \overset{\text{L.P.}}{\underset{\text{"}\frac{\infty}{\infty}\text{"}}{=}} \lim_{x\to\infty} \frac{2x}{2x} = 1$$

$\Rightarrow$ $y=1$ je vodorovná asymptota

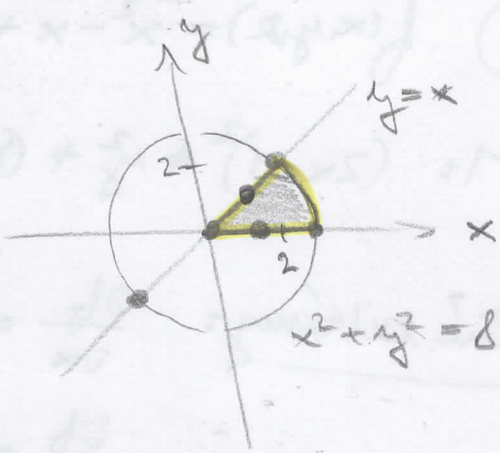
$f$ je sudá, neboť $f(-x) = \frac{(-x)^2+3}{(-x)^2+1} = \frac{x^2+3}{x^2+1} = f(x) \quad \forall x \in D_f$


Obor hodnot $\boxed{H = (1, 3\rangle}$

5.

$M: x^2+y^2 \le 8$

$0 \le y \le x$




průsečíky: $x^2+x^2=8$

$2x^2=8$

$x^2=4$

$x=\pm 2$

$f(x,y)=(x-1)^2+(y-2)^2$

$\frac{\partial f}{\partial x}=2(x-1)=0 \iff x=1$ $\quad [1,2]\notin M$

$\frac{\partial f}{\partial y}=2(y-2)=0 \iff y=2$

I. $y=x \ , \ x\in\langle 0,2\rangle$

$g(x):=f(x,x)=(x-1)^2+(x-2)^2$

$g'(x)=2(x-1)+2(x-2)=4x-6=0$

$\iff x=\frac{3}{2}$

$y=\frac{3}{2}$

II. $y=0 \ , \ x\in\langle 0,\sqrt{8}\rangle$

$g(x):=f(x,0)=(x-1)^2+4$

$g'(x)=2(x-1)=0 \iff x=1$

$y=0$

III.. oblouk kružnice: $\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} 2(x-1) & 2(y-2) \\ 2x & 2y \end{vmatrix} =$

$=4y(x-1)-4x(y-2)=-4y+8x=0 \iff \boxed{y=2x}$

$$y = 2x$$
$$x^2 + y^2 = 8$$

---

$$x^2 + 4x^2 = 8$$
$$5x^2 = 8$$
$$x^2 = \frac{8}{5}$$
$$x = \pm \frac{2\sqrt{2}}{\sqrt{5}} \qquad y = \pm \frac{4\sqrt{2}}{\sqrt{5}}$$

$$\left[ \pm \frac{2\sqrt{2}}{\sqrt{5}} ; \pm \frac{4\sqrt{2}}{\sqrt{5}} \right] \notin M$$

Kandidáti:

$[0,0]$ $\quad f(0,0) = 5$

$[2;2]$ $\quad f(2,2) = 1$

$[\sqrt{8};0]$ $\quad f(\sqrt{8},0) = (\sqrt{8}-1)^2 + 4 \doteq 7{,}34$

$[\frac{3}{2};\frac{3}{2}]$ $\quad f(\frac{3}{2},\frac{3}{2}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$[1;0]$ $\quad f(1,0) = 4$

minimum: $[\frac{3}{2};\frac{3}{2}]$

maximum: $[\sqrt{8};0]$

(6.) $f(x,y,z) = x^2 - x + 3y^2 + 2z^2 + 4z$

$M: (2x-1)^2 + y^2 + (z+1)^2 \leq 4$

<u>Volné extrémy:</u> $\frac{\partial f}{\partial x} = 2x - 1 = 0 \iff x = \frac{1}{2}$

$\frac{\partial f}{\partial y} = 6y = 0 \iff y = 0$

$\frac{\partial f}{\partial z} = 4z + 4 = 0 \iff z = -1$

$[\frac{1}{2}; 0; -1] \in M$

<u>Vázané extrémy:</u> $L(x,y,z) = f(x,y,z) + \lambda \cdot g(x,y,z)$

$L(x,y,z) = x^2 - x - 3y^2 + 2z^2 + 4z + \lambda \cdot \left((2x-1)^2 + y^2 + (z+1)^2 - 4\right)$

$\frac{\partial L}{\partial x} = 2x - 1 + \lambda \cdot (4(2x-1)) = 0 \iff (2x-1)(1+4\lambda) = 0$

$\frac{\partial L}{\partial y} = -6y + \lambda \cdot 2y = 0 \iff 2y \cdot (\lambda - 3) = 0$

$\frac{\partial L}{\partial z} = 4z + 4 + \lambda \cdot 2(z+1) = 0$

$\iff 4(z+1) + 2\lambda(z+1) = 0$

$(z+1)(4+2\lambda) = 0$

1. $\lambda = -\frac{1}{4}$: $y = 0$, $z = -1$, $(2x-1)^2 + 0^2 + 0^2 = 4$

$(2x-1)^2 - 4 = 0$

$(2x-1-2)(2x-1+2) = 0$

$(2x-3)(2x+1) = 0$

$x = \frac{3}{2} \vee x = -\frac{1}{2}$

II. $\lambda = 3$ : $x = \frac{1}{2}$, $z = -1$

$$0^2 + y^2 + 0^2 = 4$$

$$y = \pm 2$$

III. $\lambda = -2$ : $x = \frac{1}{2}$, $y = 0$

$$(z+1)^2 = 4$$

$$(z+1)^2 - 4 = 0$$

$$(z+1-2)(z+1+2) = 0$$

$$(z-1)(z+3) = 0$$

$$z = 1 \lor z = -3$$

Kandidáti:

$A = [\frac{3}{2}; 0; -1]$ $\quad f(A) = \frac{9}{4} - \frac{3}{2} + 0 + 2 - 4 = -\frac{5}{4}$

$B = [-\frac{1}{2}; 0; -1]$ $\quad f(B) = \frac{1}{4} + \frac{1}{2} + 0 + 2 - 4 = -\frac{5}{4}$

$C = [\frac{1}{2}; 2; -1]$ $\quad f(C) = \frac{1}{4} - \frac{1}{2} + 12 + 2 - 4 = \frac{39}{4}$

$D = [\frac{1}{2}; -2; -1]$ $\quad f(D) = \frac{39}{4}$

$E = [\frac{1}{2}; 0; 1]$ $\quad f(E) = \frac{1}{4} - \frac{1}{2} + 6 = \frac{23}{4}$

$F = [\frac{1}{2}; 0; -3]$ $\quad f(F) = \frac{1}{4} - \frac{1}{2} + 18 - 12 = \frac{23}{4}$

$G = [\frac{1}{2}; 0; -1]$ $\quad f(G) = \frac{1}{4} - \frac{1}{2} + 2 - 4 = -\frac{5}{4}$

maxima C, D

minima B, G, A