① $x' = 3x + 3y$
$y' = 4x - y$

$\vec{z}' = A\vec{z} \qquad A = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}$

$$|A - \lambda E| = \begin{vmatrix} 3-\lambda & 3 \\ 4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - 12 = \lambda^2 - 2\lambda - 15 =$$
$$= (\lambda - 5)(\lambda + 3) = 0$$

$\lambda_1 = 5 \quad \lambda_2 = -3$

$\lambda_1 = 5$

$(A - 5E)\vec{h}_1 = \vec{0} \qquad \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}\vec{h}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{h}_1 = (3, 2)^T$

$\lambda_2 = -3$

$(A + 3E)\vec{h}_2 = \vec{0} \qquad \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}\vec{h}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{h}_2 = (1, -2)^T$

$$\vec{z}(t) = C_1 e^{5t}\vec{h}_1 + C_2 e^{-3t}\vec{h}_2 = C_1 e^{5t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + C_2 e^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad t \in \mathbb{R}, \; C_1, C_2 \in \mathbb{R}$$

počáteční podm.

$\vec{z}(0) = (1, 0)^T$

$$\vec{z}(0) = \begin{pmatrix} 3C_1 + C_2 \\ 2C_1 - 2C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad C_1 = C_2$$
$$4C_1 = 1 \qquad C_1 = C_2 = \frac{1}{4}$$

$$\vec{z}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{4}e^{5t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \frac{1}{4}e^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4}e^{5t} + \frac{1}{4}e^{-3t} \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^{-3t} \end{pmatrix} \qquad t \in \mathbb{R}$$

(2) $x' = -7x + y$
$y' = -2x - 5y$

$\vec{z}' = A\vec{z}$ $\quad A = \begin{pmatrix} -7 & 1 \\ -2 & -5 \end{pmatrix}$

$$|A - \lambda E| = \begin{vmatrix} -7-\lambda & 1 \\ -2 & -5-\lambda \end{vmatrix} = (-7-\lambda)(-5-\lambda) + 2 =$$
$$= \lambda^2 + 12\lambda + 37 = 0$$
$$\lambda_{12} = \frac{-12 \pm \sqrt{144 - 148}}{2} = -6 \pm i$$

$\boxed{\lambda = -6 + i}$

$(A - \lambda E)\vec{h} = 0$

$$\begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix}\vec{h} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{h} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$e^{\lambda t}\vec{h} = e^{-6t} \cdot e^{it} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} = e^{-6t} \begin{pmatrix} \cos t + i \sin t \\ \cos t + i \sin t + i\cos t - \sin t \end{pmatrix} =$$

$$= \underbrace{e^{-6t} \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix}}_{Re(e^{\lambda t}h)} + i \underbrace{e^{-6t} \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix}}_{Im(e^{\lambda t}h)}$$

$$\vec{z}(t) = C_1 e^{-6t} \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + C_2 e^{-6t} \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix} \qquad t \in \mathbb{R}, \; C_1, C_2 \in \mathbb{R}$$

początkowy podm. $\vec{z}(0) = (-1, 0)^T$

$$\vec{z}(0) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \begin{matrix} \boxed{C_1 = -1} \\ C_1 + C_2 = 0 \end{matrix} \qquad \boxed{C_2 = 1}$$

$$\boxed{\vec{z}(t) = e^{-6t} \begin{pmatrix} \sin t - \cos t \\ 2\sin t \end{pmatrix} \qquad t \in \mathbb{R}}$$