Matematická analýza I, 30. 1. 2019

1. Posloupnost (a_n) je dána předpisem

$$a_n = \frac{n(3-n)}{n^2 - 43n + 240}, \quad n \in \mathbb{N}.$$

Zjistěte, pro která n je a_n větší, resp. menší, resp. rovno a_{n+1}, rozhodněte, zda je (a_n) monotónní a zda je omezená, a určete její supremum, infimum, maximum a minimum.

2. Bez použití derivací určete limitu

$$\lim \frac{(2n+1)^{6n+3}(8n-3)^{1-2n}}{(n^2-n-1)^{2n+2}}.$$

3. Určete

$$\lim_{x \to 0} \frac{\ln(\cos x - \sin x) - \ln(\cos x + \sin x)}{2^{\sqrt{1+\sin^2 x}} - 2^{\cos x}}.$$

4. Bez použití derivací určete limity funkce f dané předpisem

$$f(x) = \frac{2^{x^2} \operatorname{arctg} x}{1 - x^x}$$

v krajních bodech intervalů maximálního definičního oboru v $\mathbb{R}$.

3. $\lim\limits_{x\to 0} \dfrac{\ln(\cos x - \sin x) - \ln(\cos x + \sin x)}{2^{\sqrt{1+\sin^2 x}} - 2^{\cos x}}$

$$\overset{A.L.}{=} \underbrace{\lim_{x\to 0} \frac{\ln\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)}{\frac{\cos x - \sin x}{\cos x + \sin x} - 1}}_{=1} \cdot \lim_{x\to 0} \frac{\frac{\cos x - \sin x - \cos x - \sin x}{\cos x + \sin x}}{2^{\sqrt{1+\sin^2 x}} - 2^{\cos x}}$$

$$\downarrow$$

$$\ln(\cos x - \sin x) - \ln(\cos x + \sin x) = \ln\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$$

$$\overset{VOLSF}{=} 1 \cdot \underbrace{\lim_{x\to 0} \frac{1}{\cos x + \sin x}}_{=1} \cdot \lim_{x\to 0} \frac{-2\sin x}{2^{\cos x}\left(2^{\sqrt{1+\sin^2 x} - \cos x} - 1\right)}$$

$$\boxed{\frac{\ln y}{y-1} \xrightarrow[y\to 1]{} 1}$$

$$\overset{A.L.}{=} (-2) \cdot \underbrace{\lim_{x\to 0} \frac{1}{2^{\cos x}}}_{=\frac{1}{2}} \cdot \underbrace{\lim_{x\to 0} \frac{1}{\frac{e^{(\sqrt{1+\sin^2 x} - \cos x)\ln 2} - 1}{(\sqrt{1+\sin^2 x} - \cos x)\ln 2}}}_{=1}$$

$$\cdot \lim_{x\to 0} \frac{\sin x}{(\sqrt{1+\sin^2 x} - \cos x)\ln 2}$$

VOLSF

$$\boxed{\frac{e^R - 1}{R} \xrightarrow[R\to 0]{} 1}$$

$$= (-2) \cdot \frac{1}{2} \cdot \frac{1}{\ln 2} \cdot \lim_{x\to 0} \frac{\sin x\left(\sqrt{1+\sin^2 x} + \cos x\right)}{1 + \sin^2 x - \cos^2 x}$$

$$= -\frac{1}{\ln 2} \underbrace{\lim_{x\to 0}\left(\sqrt{1+\sin^2 x} + \cos x\right)}_{=2} \cdot \lim_{x\to 0} \frac{\sin x}{\sin^2 x + \cos^2 x + \sin^2 x - \cos^2 x}$$

$$= -\frac{2}{\ln 2} \cdot \lim_{x\to 0} \frac{\sin x}{2\sin^2 x} = -\frac{1}{\ln 2} \lim_{x\to 0} \frac{1}{\sin x}$$

$\Rightarrow$ LIMITA NEEXISTUE

④ $f(x) = \dfrac{2^{x^2}\operatorname{arctg} x}{1 - x^x}$ $\boxed{D_f = (0,1) \cup (1,\infty)}$

$$\lim_{x \to 1_\pm} \frac{2^{x^2}\operatorname{arctg} x}{1 - x^x} = \underbrace{\lim_{x \to 1} \operatorname{arctg} x}_{\frac{\pi}{4}} \cdot \underbrace{\lim_{x \to 1} 2^{x^2}}_{2} \cdot \lim_{x \to 1_\pm} \frac{1}{1 - e^{x \ln x}}$$

$$= \frac{\pi}{4} \cdot 2 \cdot \underbrace{\lim_{x \to 1_\pm} \frac{1}{\frac{1 - e^{x \ln x}}{x \ln x}}}_{= -1 \text{ nebot } \boxed{\frac{e^R - 1}{R} \xrightarrow[R \to 0]{} 1}} \cdot \lim_{x \to 1_\pm} \frac{1}{x \ln x}$$

$$= -\frac{\pi}{2} \cdot \lim_{x \to 1_\pm} \frac{1}{x \ln x} = \begin{cases} \underline{\underline{-\infty}} & \text{pro } x \to 1_+ \\ \underline{\underline{+\infty}} & \text{pro } x \to 1_- \end{cases}$$

$$\lim_{x \to 0_+} f(x) = \underbrace{\lim_{x \to 0_+} 2^{x^2}}_{1} \cdot \underbrace{\lim_{x \to 0_+} \frac{\operatorname{arctg} x}{x}}_{1} \cdot \lim_{x \to 0_+} \frac{x}{1 - x^x}$$

$$= \lim_{x \to 0_+} \frac{1}{\frac{1 - e^{x \ln x}}{x \ln x}} \cdot \lim_{x \to 0_+} \frac{x}{x \ln x}$$

$$= (-1) \cdot \frac{1}{\underbrace{\lim_{x \to 0_+} \frac{e^{x \ln x} - 1}{x \ln x}}_{\overset{(*)}{=} 1}} \cdot \lim_{x \to 0_+} \frac{1}{\ln x} = (-1) \cdot (-\infty) = \underline{\underline{+\infty}}$$

(*) $\lim\limits_{x \to 0_+} x \ln x = \lim\limits_{y \to +\infty} \frac{1}{y} \ln\left(\frac{1}{y}\right) =$

substituce: $\left| x = \frac{1}{y} \right|$

$$= \lim_{y \to \infty} \frac{\ln 1 - \ln y}{y} = -\lim_{y \to \infty} \frac{\ln y}{y} = 0$$

$$\lim_{x \to \infty} f(x) \overset{\text{A.L.}}{=} \underbrace{\lim_{x \to +\infty} \operatorname{arctg} x}_{= \frac{\pi}{2}} \cdot \lim_{x \to \infty} \frac{2^{x^2}}{1 - x^x} =$$

$$= \frac{\pi}{2} \lim_{x \to +\infty} \frac{e^{x^2 \ln 2}}{1 - e^{x \ln x}} = \frac{\pi}{2} \lim_{x \to \infty} \frac{e^{x^2 \ln 2}}{e^{x \ln x}\left(\frac{1}{e^{x \ln x}} - 1\right)}$$

$$= \frac{\pi}{2} \cdot \underbrace{\lim_{x \to \infty} \frac{1}{\frac{1}{e^{x \ln x}} - 1}}_{= -1} \cdot \lim_{x \to +\infty} e^{x^2 \ln 2 - x \ln x}$$

$$\overset{\text{VOLSF}}{=} -\frac{\pi}{2} \cdot e^{\lim\limits_{x \to +\infty}\left(x^2 \cdot \left(\ln 2 - \frac{\ln x}{x}\right)\right)}$$

$$\overset{\text{A.L.}}{=} -\frac{\pi}{2} \cdot e^{(+\infty)\cdot(\ln 2 - 0)} = -\frac{\pi}{2} \cdot (+\infty) = \underline{\underline{-\infty}}$$

$$\boxed{\frac{\ln x}{x} \xrightarrow[x \to \infty]{} 0}$$

(2) $\lim\limits_{n\to\infty} \frac{(2n+1)^{6n+3}\cdot(8n-3)^{1-2n}}{(n^2-n-1)^{2n+2}}$

$= \underbrace{\lim\limits_{n\to\infty} \frac{(2n+1)^3\cdot(8n-3)}{(n^2-n-1)^2}}_{A} \cdot \underbrace{\lim\limits_{n\to\infty} \left(\frac{(2n+1)^6}{(n^2-n-1)^2(8n-3)^2}\right)^n}_{B}$

$A = \lim\limits_{n\to\infty} \frac{\cancel{n^4}\cdot\left(2+\frac{1}{n}\right)^3\cdot\left(8-\frac{3}{n}\right)}{\cancel{n^4}\left(1-\frac{1}{n}-\frac{1}{n^2}\right)} \overset{A.L}{=} \frac{(2+0)^3\cdot(8-0)}{1-0-0}$

$= 8\cdot 8 = \underline{64}$

$B = \lim\limits_{n\to\infty} \left(\frac{\cancel{n^6}\left(2+\frac{1}{n}\right)^6}{\cancel{n^6}\left(1-\frac{1}{n}-\frac{1}{n^2}\right)^2\left(8-\frac{3}{n}\right)^2}\right)^n$

$= \lim \left(\frac{2^6\cdot\left(1+\frac{\frac{1}{2}}{n}\right)^6}{8^2\cdot\left(1-\frac{\frac{3}{8}}{n}\right)^2\cdot\left(1+\frac{-n-1}{n^2}\right)^2}\right)^n$

$= \underbrace{\left(\lim\limits_{n\to\infty}\left(1+\frac{\frac{1}{2}}{n}\right)^n\right)^6}_{\left(e^{\frac{1}{2}}\right)^6 = e^3} \cdot \frac{1}{\underbrace{\left(\lim\limits_{n\to\infty}\left(1-\frac{\frac{3}{8}}{n}\right)^n\right)^2}_{\left(e^{-\frac{3}{8}}\right)^2 = e^{-\frac{3}{4}}}}$

$\cdot \left(\lim\limits_{n\to\infty} \frac{1}{\left(1+\frac{-n-1}{n^2}\right)^n}\right)^2$

$$\lim_{n\to\infty}\left(1+\frac{-n-1}{n^2}\right)^{n} = \lim_{x\to\infty}\left(1+\frac{-x-1}{x^2}\right)^{x} =$$

HEINEHO
VĚTA

$$= \lim_{x\to+\infty} e^{x\ln\left(1+\frac{-x-1}{x^2}\right)} \underset{\text{VOLSF}}{=} e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{-x-1}{x^2}\right)}$$

$$= \dots = \underline{\underline{e^{-1}}}$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{-x-1}{x^2}\right) = \lim_{x\to\infty}\underbrace{\frac{\ln\left(1+\frac{-x-1}{x^2}\right)}{\frac{-x-1}{x^2}}}_{=1}\cdot$$

$$\cdot \lim_{x\to\infty} x\cdot\frac{-x-1}{x^2}$$

VOLSF

$$\boxed{\frac{\ln(1+R)}{R} \xrightarrow[R\to 0]{} 1}$$

$$= \lim_{x\to\infty}\frac{-x-1}{x} = \lim_{x\to\infty}\left(-1-\frac{1}{x}\right)$$

$$= \underline{\underline{-1}}$$

$$\Rightarrow B = e^{3}\cdot\frac{1}{e^{-\frac{3}{4}}}\cdot\left(\frac{1}{e^{-1}}\right)^{2} = e^{3}\cdot e^{\frac{3}{4}}\cdot e^{2} = e^{3+\frac{3}{4}+2}$$

$$= e^{\frac{23}{4}}$$

$$\Rightarrow \lim_{n\to\infty} a_n = A\cdot B = \underline{\underline{64e^{\frac{23}{4}}}}$$