6. minitest RMF

Varianta A

8. 11. 2024

Určete limitu posloupnosti funkcí $f_n(x)$ v $\mathcal{D}'(\mathbb{R})$ a rozhodněte, zda existuje i v klasickém smyslu.

$$f_n(x) = \frac{2n^2+3n+1}{n^3x^2-4n^2x+13n}$$

$$(f_n, \varphi) = \int_{\mathbb{R}} \frac{2n^2+3n+1}{n^3x^2-4n^2x+13n}\varphi(x)\,dx = \int_{\mathbb{R}} \frac{2n^2+3n+1}{n(t^2-4t+13)}\varphi\left(\frac{t}{n}\right)\frac{dt}{n}$$

$$\left| \begin{array}{l} nx = t \\ n\,dx = dt \\ dx = \frac{dt}{n} \end{array} \right|$$

$$= \int_{\mathbb{R}} \frac{2n^2+3n+1}{n^2}\cdot\frac{1}{(t-2)^2+9}\varphi\left(\frac{t}{n}\right)dt =$$

$$= \frac{1}{9}\int_{\mathbb{R}}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\cdot\frac{1}{\left(\frac{t-2}{3}\right)^2+1}\cdot\varphi\left(\frac{t}{n}\right)dt \xrightarrow[n\to\infty]{} \frac{2}{9}\varphi(0)\cdot\int_{\mathbb{R}}\frac{1}{\left(\frac{t-2}{3}\right)^2+1}dt$$

LEBESGUEOVA VĚTA

$$\forall n\in\mathbb{N}\ \forall t\in\mathbb{R}: \left|\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\cdot\frac{1}{\left(\frac{t-2}{3}\right)^2+1}\cdot\varphi\left(\frac{t}{n}\right)\right| \leq \underbrace{\frac{6\max_{x\in\mathbb{R}}\varphi(x)}{\left(\frac{t-2}{3}\right)^2+1}}\ \in L^1(\mathbb{R})$$

$$\int_{\mathbb{R}}\frac{1}{\left(\frac{t-2}{3}\right)^2+1}dt = \left| \begin{array}{l} \frac{t-2}{3} = z \\ \frac{1}{3}dt = dz \\ dt = 3dz \end{array}\right| = \int_{\mathbb{R}}\frac{3\,dz}{z^2+1} = 3\left[\operatorname{arctg} z\right]_{-\infty}^{\infty}$$

$$= 3\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = 3\pi$$

tedy: $(f_n,\varphi) \xrightarrow[n\to\infty]{} \frac{2}{9}\varphi(0)\cdot 3\pi = \underline{\underline{\frac{2\pi}{3}\delta(x)}}$

$$f_n(0) = \frac{2n^2+3n+1}{13n} = \frac{2}{13}n+\frac{3}{13}+\frac{1}{13n} \xrightarrow[n\to\infty]{} \infty \Rightarrow \text{nemá limitu v klasickém smyslu}$$

6. minitest RMF

Varianta B

8. 11. 2024

Určete limitu posloupnosti funkcí $f_n(x)$ v $\mathcal{D}'(\mathbb{R})$ a rozhodněte, zda existuje i v klasickém smyslu.

$$f_n(x) = \frac{3n^2+2n+1}{n^3x^2-6n^2x+13n}$$

$$(f_n,\varphi) = \int_{\mathbb{R}} \frac{3n^2+2n+1}{n}\cdot\frac{1}{n^2x^2-6nx+13}\cdot\varphi(x)\,dx =$$

$$\underset{\left|\begin{matrix} nx=t \\ n\,dx=dt \\ dx=\frac{1}{n}dt\end{matrix}\right|}{=} \int_{\mathbb{R}} \left(3n+2+\frac{1}{n}\right)\cdot\frac{1}{t^2-6t+13}\cdot\varphi\left(\frac{t}{n}\right)\frac{1}{n}\,dt$$

$$= \int_{\mathbb{R}} \underbrace{\left(3+\frac{2}{n}+\frac{1}{n^2}\right)\cdot\frac{1}{(t-3)^2+4}\cdot\varphi\left(\frac{t}{n}\right)}_{=:\,g_n(t)}\,dt \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \frac{3\varphi(0)}{(t-3)^2+4}\,dt$$

LEBESGUEOVA VĚTA

$$\forall n\in\mathbb{N}\ \forall t\in\mathbb{R}:\ |g_n(t)| \le \frac{6\max_{x\in\mathbb{R}}\varphi(x)}{(t-3)^2+4} \in L^1(\mathbb{R})$$

$$\int_{\mathbb{R}} \frac{1}{(t-3)^2+4}\,dt = \frac{1}{4}\int_{\mathbb{R}} \frac{1}{\left(\frac{t-3}{2}\right)^2+1}\,dt = \frac{1}{4}\cdot 2\underbrace{\left[\operatorname{arctg}\left(\frac{t-3}{2}\right)\right]_{-\infty}^{\infty}}_{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi} = \frac{\pi}{2}$$

tedy: $(f_n,\varphi) \xrightarrow[n\to\infty]{} 3\varphi(0)\cdot\frac{\pi}{2} = \frac{3\pi}{2}\delta(x)$

$$f_n(0) = \frac{3n^2+2n+1}{13n} = \frac{3}{13}n+\frac{2}{13}+\frac{1}{13n} \xrightarrow[n\to\infty]{} \infty$$

$\Rightarrow$ f_n nemají limitu v klasickém smyslu